Springer

Journal of Mechanical Science and Technology

Journal of Mechanical Science and Technology 23 (2009) 768~774

www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-008-1212-0

Investigation on the turbulent stress anisotropy of axisymmetric turbulence under rapid rotation †

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(Manuscript Received April 29, 2008; Revised November 5, 2008; Accepted December 14, 2008)

Abstract

The root cause of different oscillatory behavior of turbulent stress anisotropy under rapid rotation of initially axisymmetric turbulence is theoretically investigated. For this, based on the velocity spectral tensor of axisymmetric turbulence, the rapid part of the pressure-strain is determined and the equation of the turbulent stress anisotropy is solved for initial conditions generated by axisymmetric expansion and contraction of isotropic turbulence. As is well known, the damping of turbulent stress anisotropy under rapid rotation is observed for both initial conditions, and this feature is attributed to the linear rapid rotation effect on turbulence. On the other hand, the oscillatory development of turbulent stress anisotropy can be seen conspicuously only for the initial turbulence generated by axisymmetric expansion. This selective oscillatory feature is found to be strongly related to the total strain that is applied to the isotropic turbulence to generate the initial axisymmetric turbulence. And, through an asymptotic approach, it is also found that the material frame-indifference principle of two-dimensional turbulence is the underlying physics in this different oscillatory behavior.

Keywords: Turbulence simulation; Homogeneous turbulence; Rotating flow; Material frame-indifference; Two-dimensional turbulence

1. Introduction

It is well known that one of the most noticeable rapid rotation effects on initially anisotropic turbulence is the damped oscillation of stress anisotropy of turbulence due to phase scrambling [1, 2]; and any conventional model of rapid pressure-strain based on the closure scheme with Reynolds stress tensor only cannot cope with this physics [3]. The damped oscillation of turbulent stress anisotropy due to rapid rotation has been well manifested for an initially weakly anisotropic model spectrum [1, 2] and an initial anisotropic spectrum generated by the plain strain of isotropic turbulence [2, 4]. This physics was also demonstrated numerically for an initial anisotropic spectrum generated by axisymmetric expansion of isotropic turbulence [4]. Kassinos and Reynolds [5] investigated the rapid rotation effect on the axisymmetric turbulence. They studied the behavior of the turbulent stress anisotropy due to the rapid rotation for an initially anisotropic turbulence generated by an axisymmetric contraction of isotropic turbulence. Their result shows clear damping of stress anisotropy. However, unlike previous studies [1, 2, 4], oscillation of the turbulent stress anisotropy is hardly noticeable. To investigate the reason for such discrepancy and to find the underlying physics, Kassinos and Reynolds' [5] work for axisymmetric turbulence is extended in the present study, including both cases of initial conditions generated by axisymmetric contraction or axisymmetric expansion of isotropic turbulence.

† This paper was recommended for publication in revised form by Associate Editor Yang Na
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2. Analytic solutions for rapid rotating homogeneous turbulence

The rotating homogeneous turbulence is an anisotropic turbulence with a rotational axis as its axis of symmetry. In this respect, the velocity spectral tensor suggested by Cambon and Jacquin [6] for a general anisotropic turbulence subjected to the rotation can be a good starting point. In the present study, however, we use the velocity spectral tensor suggested by Kassinos and Reynolds [5] instead, considering the axisymmetric turbulence is the simplest form of rotating homogeneous turbulence for which any theoretical treatment is possible.

Applying the invariant theory of turbulence with homogeneity and incompressibility conditions, the velocity spectral tensor $\Phi_{ij}(\vec{\kappa})$ of axisymmetric turbulence with an additional reflectional symmetry in the plane normal to $\vec{\lambda}$ was determined by [5]

$$\begin{aligned}\Phi_{ij}(\vec{\kappa}) &= \frac{\mathcal{E}(\kappa,\alpha)}{4\pi\kappa^2}P_{ij}(\vec{\kappa}) \\ &+C_5(\kappa,\alpha)\left[\frac{\alpha^2}{\kappa^4}\kappa_i\kappa_j - \frac{\alpha}{\kappa^2}\kappa_i\lambda_j - \frac{\alpha}{\kappa^2}\kappa_j\lambda_i\right] \\ &+C_5(\kappa,\alpha)\left[\lambda_i\lambda_j - \frac{1}{2}\left(1-\frac{\alpha^2}{\kappa^2}\right)P_{ij}(\vec{\kappa})\right] \\ &+\kappa C_9(\kappa,\alpha)\varepsilon_{inm}\left(\frac{1}{\kappa}\kappa_n\lambda_m\lambda_j - \frac{\alpha}{\kappa^3}\kappa_n\lambda_m\kappa_j\right) \\ &+\kappa C_9(\kappa,\alpha)\varepsilon_{jnm}\left(\frac{1}{\kappa}\kappa_n\lambda_m\lambda_i - \frac{\alpha}{\kappa^3}\kappa_n\lambda_m\kappa_i\right) \qquad (1)\end{aligned}$$

where $\vec{\kappa}$ is wavenumber vector with $\kappa = \sqrt{\kappa_i\kappa_i}$, $\vec{\lambda}$ is symmetry axis with $\lambda_i\lambda_i = 1$, $\alpha = \kappa_i\lambda_i$, $P_{ij}(\vec{\kappa})(=\delta_{ij} - \kappa_i\kappa_j/\kappa^2)$ is a projection operator, and ε_{ijk} is a permutation symbol. Here, $\mathcal{E}(\kappa,\alpha)$ and $C_5(\kappa,\alpha)$ are even functions of α, and $C_9(\kappa,\alpha)$ is an odd function of α. Since the rapid part of pressure-strain in the Reynolds stress equation at a reference frame rotating with a rate $\vec{\Omega}$ is given by

$$\begin{aligned}\Pi_{ij}^{Rapid,\Omega} &= 2\left(\int \Phi_{il}(\vec{\kappa})\frac{\kappa_j\kappa_k}{\kappa^2}d\vec{\kappa}\right)(\varepsilon_{kml}\Omega_m) \\ &+2\left(\int \Phi_{jl}(\vec{\kappa})\frac{\kappa_i\kappa_k}{\kappa^2}d\vec{\kappa}\right)(\varepsilon_{kml}\Omega_m), \qquad (2)\end{aligned}$$

substituting Eq. (1) into Eq. (2) gives

$$\begin{aligned}\Pi_{ij}^{Rapid,\Omega} &= -4\pi\int_0^\infty\int_0^\pi \kappa^3 C_9(\kappa,\alpha)\sin^5\phi\cos\phi\, d\phi\, d\kappa \\ &\times\Omega(\delta_{ij} - 3\lambda_i\lambda_j). \qquad (3)\end{aligned}$$

Note that zero mean strain is assumed and a coordinate system in which κ_1 axis coincides with the rotation axis is used to derive the above result. Since we are concerned about the rapid rotation effect, the above equation can be simplified with the rapid distortion theory. Under a rapid distortion limit, the governing equation of the velocity spectral tensor is given by [7]

$$\begin{aligned}\frac{\partial\Phi_{ij}(\vec{\kappa})}{\partial t} &= -2\varepsilon_{klm}\Omega_m P_{il}(\vec{\kappa})\Phi_{kj}(\vec{\kappa}) \\ &-2\varepsilon_{klm}\Omega_m P_{jl}(\vec{\kappa})\Phi_{ik}(\vec{\kappa}) \qquad (4)\end{aligned}$$

and substitution of Eq. (1) in Eq. (4), just as done by Kassinos and Reynolds [5], gives the following equations:

$$\frac{\partial\mathcal{E}}{\partial t} = 0,\quad \frac{\partial C_5}{\partial t} = 8\Omega\frac{\alpha}{\kappa}(\kappa C_9) \quad \text{and}$$

$$\frac{\partial(\kappa C_9)}{\partial t} = -2\Omega\frac{\alpha}{\kappa}C_5.$$

Their solutions are found to be

$$\mathcal{E}(\kappa,\alpha,t) = \mathcal{E}(\kappa,\alpha,0),$$

$$C_5(\kappa,\alpha,t) = C_5(\kappa,\alpha,0)\cos 4\Omega\frac{\alpha}{\kappa}t \quad \text{and}$$

$$\kappa C_9(\kappa,\alpha,t) = -\frac{1}{2}C_5(\kappa,\alpha,0)\sin 4\Omega\frac{\alpha}{\kappa}t. \qquad (5)$$

Considering the generation mechanism of axisymmetric turbulence by straining an isotropic turbulence, $C_9(\kappa,\alpha,0)$ is assumed to be zero in the above solutions because it represents the breaking of reflectional symmetry that is unlikely to be present at the initial instant of generation. If $\partial C_5/\partial t = 8\Omega\alpha/\kappa(\kappa C_9)$ is substituted in Eq. (3), it becomes

$$\begin{aligned}\Pi_{ij}^{Rapid,\Omega} &= \frac{\partial}{\partial t}\left[\frac{3\pi}{2}\int_0^\infty\int_0^\pi \kappa^2 C_5(\kappa,\alpha,t)\sin^5\phi\, d\phi\, d\kappa\right] \\ &\times\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right). \qquad (6)\end{aligned}$$

One can assume that an initial axisymmetric turbulence was generated by an axisymmetric strain of isotropic turbulence at $t=-t_s$ just as in Kassinos and Reynolds [5]. Then the integral of the above equation can be rewritten as follows (See, Appendix):

$$\frac{3\pi}{2}\int_0^\infty\int_0^\pi \kappa^2 C_5(\kappa,\alpha,t)\sin^5\phi\, d\phi\, d\kappa = \frac{3K_s}{8}(c^{-2}-c) \times\mathcal{F}(t) \tag{7}$$

where K_s is the turbulent kinetic energy at $t=-t_s$, $c[\equiv \exp(\int_{-t_s}^{0} S dt')]$ is the total strain rate in axial direction and $\mathcal{F}(t)$ is given as

$$\mathcal{F}(t) \equiv \int_{-1}^{+1} \frac{(1-x^2)^2\cos(4\Omega t x)}{\left[c^{-1}+(c^2-c^{-1})x^2\right]^{3/2}}dx. \tag{8}$$

Note that S is the axial strain rate and $x\equiv\cos\phi$. Combining Eqs. (6) with (7) gives the analytic form of the rapid pressure-strain under the rapid distortion limit as

$$\Pi_{ij}^{Rapid,\Omega} = \frac{3K_s}{8}(c^{-2}-c)\frac{\partial\mathcal{F}(t)}{\partial t}(\lambda_i\lambda_j-\frac{1}{3}\delta_{ij}). \tag{9}$$

The above equation can be further simplified in terms of the well-known turbulent stress anisotropy tensor $b_{ij}[\equiv \tau_{ij}/(2K)-1/3\delta_{ij}]$ where $K(\equiv 0.5\tau_{ii})$ is the turbulent kinetic energy and $\tau_{ij}(\equiv \int \Phi_{ij} d\vec{\kappa})$ is the Reynolds stress tensor. Using Eq. (1), b_{ij} can be written as

$$2Kb_{ij} = \left[-\frac{1}{4}\int_0^\infty\int_0^\pi \mathcal{E}(\kappa,\alpha,t)\sin\phi(2-3\sin^2\phi)d\phi d\kappa + \frac{3\pi}{2}\int_0^\infty\int_0^\pi \kappa^2 C_5(\kappa,\alpha,t)\sin^5\phi\, d\phi\, d\kappa\right] \times\left(\lambda_i\lambda_j-\frac{\delta_{ij}}{3}\right) \tag{10}$$

and as a result, it is follows that

$$2K(b_{ij,0}-b_{ij,\infty}) = \frac{3K_s}{8}(c^{-2}-c)\mathcal{F}(0) \times\left(\lambda_i\lambda_j-\frac{1}{3}\delta_{ij}\right) \tag{11}$$

where subscript '0' represents initial instant and '∞' represents the time at which, considering the rapid distortion solution for $C_5(\kappa,\alpha,t)$, the second integral of Eq. (10) vanishes as $4\Omega t\to\infty$. Applying Eq. (11) to Eq. (9) gives the representation of the rapid part of pressure-strain in terms of b_{ij} as

$$\frac{\Pi_{ij}^{Rapid,\Omega}}{2K} = \frac{\left(b_{ij,0}-b_{ij,\infty}\right)}{\mathcal{F}(0)}\frac{\partial\mathcal{F}(t)}{\partial t}. \tag{12}$$

Introducing the anisotropy tensor of structure dimensionality [3] $y_{ij}[\equiv Y_{ij}(2K)-1/3\delta_{ij}]$ that is determined by using Eq. (1)

$$2Ky_{ij} = \frac{1}{2}\int_0^\infty\int_0^\pi \mathcal{E}(\kappa,\alpha,t)\sin\phi(2-3\sin^2\phi)d\phi d\kappa \times\left(\lambda_i\lambda_j-\frac{1}{3}\delta_{ij}\right), \tag{13}$$

Eq. (12) is also rewritten as

$$\frac{\Pi_{ij}^{Rapid,\Omega}}{2K} = \frac{(b_{ij,0}+\frac{1}{2}y_{ij,0})}{\mathcal{F}(0)}\frac{\partial\mathcal{F}(t)}{\partial t} \tag{14}$$

where a condition of $b_{ij,\infty}=-1/2y_{ij,0}$ is used. Note that $Y_{ij}(\equiv \int \kappa_i\kappa_j/\kappa^2\Phi_{kk}d\vec{\kappa})$ is the structure dimensionality tensor [3]. Equation (14) is the exact representation of the rapid part of pressure-strain under the rapid distortion limit.

Specific values of $b_{ij,0}$ and $1/2y_{ij,0}$ can be obtained by applying the same procedure of Appendix to Eqs. (10) and (13), respectively. They are given in terms of total strain rate as

$$b_{ij,0} = \left[\frac{-\frac{1}{4}\int_{-1}^{1}\frac{\left[\left(c^2+c^{-1}\right)+\left(c^2-c^{-1}\right)x^2\right]\left(3x^2-1\right)}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx}{\int_{-1}^{1}\frac{\left[\left(c^2+c^{-1}\right)+\left(c^2-c^{-1}\right)x^2\right]}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx} + \frac{\frac{3}{4}\left(c^{-1}-c^2\right)\int_{-1}^{1}\frac{\left(1-x^2\right)^2}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx}{\int_{-1}^{1}\frac{\left[\left(c^2+c^{-1}\right)+\left(c^2-c^{-1}\right)x^2\right]}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx}\right] \times\left(\lambda_i\lambda_j-\frac{1}{3}\delta_{ij}\right) \tag{15}$$

$$\frac{1}{2}y_{ij,0} = \frac{\frac{1}{4}\int_{-1}^{1}\frac{\left[\left(c^2+c^{-1}\right)+\left(c^2-c^{-1}\right)x^2\right]\left(3x^2-1\right)}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx}{\int_{-1}^{1}\frac{\left[\left(c^2+c^{-1}\right)+\left(c^2-c^{-1}\right)x^2\right]}{\left[c^{-1}+\left(c^2-c^{-1}\right)x^2\right]^{3/2}}dx}$$

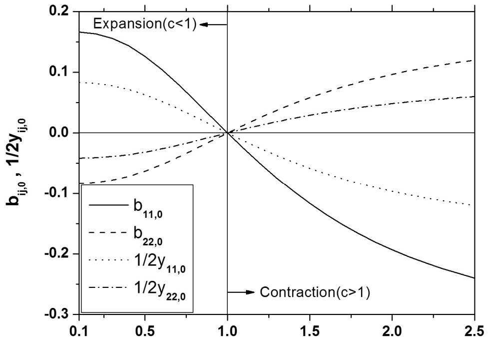

Fig. 1. Initial anisotropies of b_{ij} and y_{ij} depending on total axial strain rate.

$$\times\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right) \tag{16}$$

Fig. 1 shows the variations of $b_{ij,0}$ and $1/2y_{ij,0}$ depending on the total strain rate c from 0.1 to 2.5. Here $c > 1$ and $c < 1$ cases are applicable to axisymmetric contraction and axisymmetric expansion of isotropic turbulence, respectively. Note that the value of $b_{ij,0}$ obtained in the present study is identical to that of Lee [8], but its representation is different from Lee [8] because the velocity spectral tensor in physical coordinate is used in the present study (see Appendix).

3. Results and discussions

From now on, using Eq. (14), the behavior of the turbulent stress anisotropy tensor under the rapid rotation is to be investigated.

Integrating Eq. (4) over the whole Fourier space and rearranging leads to non-zero b_{ij} equation for rapidly rotating turbulence as follows:

$$\frac{\partial b_{11}}{\partial t} = \frac{1}{2K}\Pi_{11}^{Rapid,\Omega} \quad \text{and}$$
$$\frac{\partial b_{22}}{\partial t} = \frac{\partial b_{33}}{\partial t} = \frac{1}{2K}\Pi_{22}^{Rapid,\Omega}.$$

And the solution is found to be

$$b_{ij}(t^*) = \left(b_{ij,0} + \frac{1}{2}y_{ij,0}\right)\frac{\mathcal{F}(t^*)}{\mathcal{F}(0)} - \frac{1}{2}y_{ij,0} \tag{17}$$

where $t^* \equiv 4\Omega t$.

Numerical integration of Eq. (17) for $0.1 \le c \le 2.5$ is shown in Fig. 2 with $\Delta c = 0.1$ difference. For axisymmetric expansion case of $0.1 \le c < 1.0$, the oscillatory damping behavior of $b_{ij}(t^*)$ is found as t^* increases.

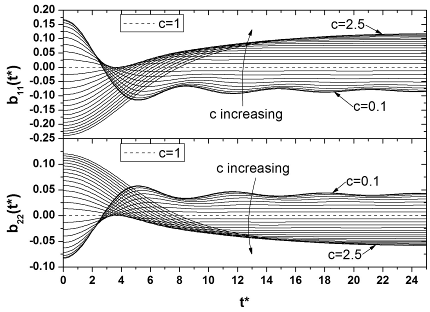

Fig. 2. Behavior of turbulent stress anisotropy b_{ij} for $0.1 \le c \le 2.5$ with $\Delta c = 0.1$.

b_{ij} is damped to $-0.5y_{ij,0}$ which is about half of the absolute magnitude of $b_{ij,0}$ (see, Fig. 1), and this damping accompanies oscillatory motion which is intensified as initial anisotropy increases. For axisymmetric contraction case of $1.0 < c \le 2.5$, one can also find the damping of b_{ij}. However, oscillation of b_{ij} is hardly observed except for very slight oscillatory motion near $c = 1$. These observations explain why the oscillatory behaviors are found differently between the results of Cambon, Jacquin and Lubrano [4] for $c < 1$ case (see their Fig. 2(a)) and that of Kassinos and Reynolds [5] for $c > 1$ case (see their Fig. 2 in appendix K), although they are produced by the same axisymmetric turbulence except different c values. To be more specific, considering Eq. (17) and (8), one can notice that the common damping feature of b_{ij} irrespective of c is due to the decrease of period of cosine function in Eq. (8) as $t^* \to \infty$. Since the linear solution of the present study, which is Eq. (5), includes this very cosine function, the common damping feature can be attributed to the well-known rapid rotation effect on turbulence. That is, the phase scrambling [1, 2] or the scrambling of the initial polarization [4]. As for the origin of difference in the oscillatory motion of b_{ij} depending on c, one needs to do more scrutiny. For this, consider two extreme cases of total strain rate.

Consider $c \to 0$ case, first. This is the extreme case of axisymmetric expansion. As $c \to 0$, $b_{ij,0}$ and $1/2y_{ij,0}$ are approximated as follows:

$$b_{ij,0} \approx \frac{1}{4}\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right) \quad \text{and}$$

$$\frac{1}{2}y_{ij,0} \approx \frac{1}{8}\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right) \tag{18}$$

Note that all c^2 terms in Eq. (15) and (16) are neglected in deriving the above results. Similarly, $\mathcal{F}(t^*)/\mathcal{F}(0)$ is approximated as follows. As $c \to 0$,

$$\frac{\mathcal{F}(t^*)}{\mathcal{F}(0)} \approx \frac{\displaystyle\int_{-1}^{1} \frac{\left(1-x^2\right)^2 \cos(t^* x)}{\left(c^{-1}-c^{-1}x^2\right)^{3/2}} dx}{\displaystyle\int_{-1}^{1} \frac{\left(1-x^2\right)^2}{\left(c^{-1}-c^{-1}x^2\right)^{3/2}} dx} = 2\frac{J_1(t^*)}{t^*} \tag{19}$$

where $J_1(t^*)$ is the Bessel function of the first kind of order one.

Next, consider $c \to \infty$ case. This corresponds to the extreme case of axisymmetric contraction. As $c \to \infty$, $b_{ij,0}$ and $1/2y_{ij,0}$ are approximated as follows:

$$b_{ij,0} \approx -\frac{1}{2}\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right) \text{ and}$$
$$\frac{1}{2}y_{ij,0} \approx -\frac{1}{4}\left(\lambda_i\lambda_j - \frac{1}{3}\delta_{ij}\right) \tag{20}$$

Note that all c^{-1} terms are neglected except that c^{-1} in denominators of integrand of Eq. (15) and (16) that is unrelated to x are preserved in deriving these results. Likewise, $\mathcal{F}(t^*)/\mathcal{F}(0)$ can be approximated for $c \to \infty$,

$$\frac{\mathcal{F}(t^*)}{\mathcal{F}(0)} \approx \frac{\displaystyle\int_{-1}^{1} \frac{\left(1-x^2\right)^2 \cos(t^* x)}{\left(c^{-1}+c^2x^2\right)^{3/2}} dx}{\displaystyle\int_{-1}^{1} \frac{\left(1-x^2\right)^2}{\left(c^{-1}+c^2x^2\right)^{3/2}} dx} \approx 1. \tag{21}$$

As one can see from Eq. (21), the integrand except cosine term is a banded function around $x = 0$ and it gets narrower as $c \to \infty$. Hence, the oscillatory nature of cosine becomes ineffective as $c \to \infty$, and as a result, any oscillatory behavior of $\mathcal{F}(t^*)/\mathcal{F}(0)$ is suppressed as $c \to \infty$. Meanwhile, oscillatory behavior of $\mathcal{F}(t^*)/\mathcal{F}(0)$ is well developed as $c \to 0$ because the integrand of Eq. (19) approaches a well defined function for $-1 \le x \le 1$ as $c \to 0$. These findings are compatible with the oscillatory feature of b_{ij} depending on the total strain rates.

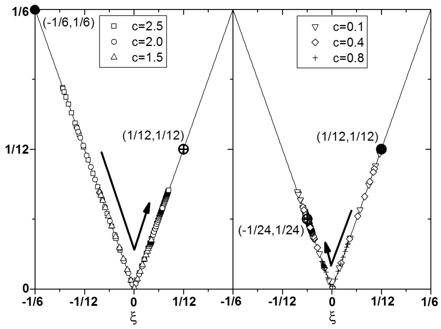

Fig. 3. Evolution of turbulent stress anisotropy on anisotropy invariant map for selected c. (state points • are the initial state and ⊕ are the final state.)

Before conclusions, it is worthwhile to mention the physical implication of $\mathcal{F}(t^*)/\mathcal{F}(0) \approx 1$ in Eq. (21) as $c \to \infty$. Fig. 3 shows the invariant maps of b_{ij} for the axisymmetric contraction case. The axisymmetric expansion case is also included in the figure. $\xi \equiv \left(b_{ij}b_{jk}b_{ki}/6\right)^{1/3}$ and $\eta \equiv \left(b_{ij}b_{ji}/6\right)^{1/2}$ are used for coordinate axes. As can be seen in the figure, an initial and a final point on the invariant map approach asymptotically particular points such as $(-1/6,\ 1/6)$ and $(1/12,\ 1/12)$, respectively as $c \to \infty$. However, when the initial state is at $(-1/6,\ 1/6)$ exactly, an initially axisymmetric turbulence becomes an exact two-dimensional turbulence because $y_{11,0} = -1/3$ at this state [1]. So, according to the material frame-indifference of two-dimensional turbulence [3, 5], the initial turbulence state remains stagnant, and as a corollary, $\mathcal{F}(t^*)/\mathcal{F}(0)$ becomes '1' irrespective of t^*. Eq. (17) really shows that $b_{ij}(t^*) = b_{ij,0}$ when $\mathcal{F}(t^*)/\mathcal{F}(0) = 1$. Note that if the initial turbulence state is different infinitesimally from the exact two dimensional state, the turbulence state will move to $(1/12,\ 1/12)$ but it never reaches the exact point. On the other hand, an initial and a final point on the invariant map also approach particular points such as $(1/12,\ 1/12)$ and $(-1/24,\ 1/24)$, respectively as $c \to 0$. However, unlike the axisymmetric contraction case, the final turbulence state moves around $(-1/24,\ 1/24)$, and eventually it reaches this very point when the initial point is at $(1/12,\ 1/12)$ exactly as $c \to 0$. This is because turbulence does not have any special character such as two-dimensionality at $(1/12,\ 1/12)$.

4. Conclusions

The analytic representation of the rapid part of the pressure strain was obtained for the axisymmetric rotating homogeneous turbulence under rapid rotation. Combining it with the linearized equation of the turbulence anisotropy tensor, the temporal variation of the turbulence stress anisotropy under the rapid rotation was investigated.

Through analysis, it was found that the damping of initial turbulence stress anisotropy is a common feature of rapidly rotating turbulence for both initial turbulences generated by axisymmetric contraction and axisymmetric expansion of an isotropic turbulence due to the linear effect of rotation.

Meanwhile, the oscillatory feature of the turbulence stress anisotropy was found to be different depending on the total strain applied to generate initial axisymmetric turbulence. As for this, through the asymptotic approach, it has been verified that two-dimensionality of turbulence established at the infinity rate of the total strain satisfies the material frame-indifference principle under rapid rotation as a result the oscillating motion of the stress anisotropy is completely suppressed. However, the turbulence never reaches the two-dimensional state and the oscillatory motion in the stress anisotropy is not suppressed either when the total strain approaches to zero. This finding implies that the dimensionality of turbulence is a factor for governing the oscillatory motion of the anisotropy of turbulence in the rotating homogeneous turbulence and explains the discrepancy found among previous researches.

In reflection of the present result, any turbulence model to resolve the damping of the turbulence stress anisotropy should reproduce the linear effect of rapid rotation faithfully, and it also should be compatible with the material frame-indifference principle to reproduce the exact oscillatory motions of stress anisotropy. In this respect, our present analytic result can be used as a benchmark test problem for the development of any rotating turbulence model.

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Appendix: Derivation of Eq. (7)

If an isotropic turbulence at $t = -t_s$ is strained axisymmetrically until $t = 0$, then the velocity correlation tensor R_{11} at $t = 0$ is given in the strained coordinate as [8]

$$R_{11}(\vec{r},0) \equiv \int \Phi_{11}(\vec{\tilde{\kappa}},0) e^{i\vec{\tilde{\kappa}}\bullet\vec{\tilde{r}}} d\vec{\tilde{\kappa}} = \int \frac{E_s(\tilde{\kappa})}{4\pi} \frac{c^{-2}\tilde{\kappa}_{23}^2}{\left(c^{-3}\tilde{\kappa}_1^2 + \tilde{\kappa}_{23}^2\right)^2} e^{i\vec{\tilde{\kappa}}\bullet\vec{\tilde{r}}} d\vec{\tilde{\kappa}} \quad \text{(A1)}$$

where $E_s(\tilde{\kappa})$ is the spectrum function of isotropic turbulence at $t = -t_s$, $\tilde{\kappa}_{23}^2 \equiv \tilde{\kappa}_2^2 + \tilde{\kappa}_3^2$ and '~' represents a variable in the strained coordinate. Here, the total strain vector $\vec{e} = (e_1, e_2, e_3)$ defined by $e_\alpha \equiv Exp(\int_{-t_s}^{0} S_{\alpha\alpha} dt')$ (no summation in Greek indices) is used with the axisymmetric strain rate of $S_{11} = S$ and $S_{22} = S_{33} = -S/2$. Therefore, if $e_1 = c$ then, $e_2 = e_3 = 1/\sqrt{c}$. However, the strained coordinate is related to the physical coordinate through $\tilde{r}_\alpha = r_\alpha / e_\alpha$ and $\tilde{\kappa}_\alpha = e_\alpha \kappa_\alpha$ where r_α and κ_α are a separation vector and a wavenumber vector in physical coordinate, respectively. Therefore, change of variables on the right hand side of Eq. (A1) in terms of r_i and κ_i gives

$$R_{11}(\vec{r},0) = \int \frac{E_s(\kappa')}{4\pi\kappa^2} \frac{c^{-1}\kappa_{23}^2}{\kappa^2} e^{i\vec{\kappa}\bullet\vec{r}} d\vec{\kappa} \quad \text{(A2)}$$

where $\kappa' \equiv \sqrt{c^2\kappa_1^2 + c^{-1}\kappa_{23}^2}$. Furthermore, R_{11} in physical coordinate is also given by

$$R_{11}(\vec{r},0) = \int \Phi_{11}(\vec{\kappa},0) e^{i\vec{\kappa}\bullet\vec{r}} d\vec{\kappa}. \tag{A3}$$

Since $\overline{u_1^2}(0) = R_{11}(\vec{\tilde{r}},0)|_{\vec{\tilde{r}}=0} = R_{11}(\vec{r},0)|_{\vec{r}=0}$, equating Eq. (A2) and (A3) for $\vec{\tilde{r}} = \vec{r} = 0$ gives the initial velocity spectral tensor at the physical coordinate as

$$\Phi_{11}(\vec{\kappa},0) = \frac{E_s(\kappa')}{4\pi\kappa^2}\frac{c^{-1}\kappa_{23}^2}{\kappa^2}. \tag{A4}$$

In a similar manner, $\Phi_{22}(\vec{\kappa},0)$ and $\Phi_{33}(\vec{\kappa},0)$ in the physical coordinate can also be determined. Since the velocity spectral tensor components are determined in the physical coordinate, direct substitution of determined $\Phi_{ij}(\vec{\kappa},0)$ into $C_5(\kappa,\alpha,0)$ derived from Eq. (1) in terms of $\Phi_{ij}(\vec{\kappa},0)$ gives an equation as

$$C_5(\kappa,\alpha,0) \equiv \frac{E_s(\kappa')}{4\pi}\frac{\left(c^{-2}-c\right)}{c^2\kappa_1^2 + c^{-1}\kappa_{23}^2}. \tag{A5}$$

Using the above $C_5(\kappa,\alpha,0)$, then, Eq. (7) is determined by

$$\frac{3\pi}{2}\int_0^\infty\int_0^\pi \kappa^2 C_5(\kappa,\alpha,0)\cos\left(4\Omega\frac{\alpha}{\kappa}t\right)\sin^5\phi\, d\phi\, d\kappa =$$
$$\frac{3K_s}{8}\left(c^{-2}-c\right)\int_{-1}^{1}\frac{\left(1-x^2\right)^2\cos(4\Omega tx)}{\left[c^{-1}+\left(c^2-c^{-1}\right)\right]^{3/2}}dx \tag{A6}$$

where change of variables such as $\alpha = \kappa_1 = \kappa\cos\phi$ and $\cos\phi = x$, and an integral of

$$\int_0^\infty E_s(\kappa')d\kappa = \frac{K_s}{\sqrt{c^{-1}+\left(c^2-c^{-1}\right)x^2}}$$

are used. Note that K_s is the turbulent kinetic energy of isotropic turbulence at $t = -t_s$.

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