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The unified equations to obtain the exact solutions of piezoelectric plane beam subjected to arbitrary loads †

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Abstract

The unified equations to obtain the exact solutions for piezoelectric plane beam subjected to arbitrary mechanical and electrical loads with various ends supported conditions is founded by solving functional equations. Comparing this general method with traditional trial-and-error method, the most advantage is it can obtain the exact solutions directly and does not need to guess and modify the form of stress function or electric displacement function repeatedly. Firstly, the governing equation for piezoelectric plane beam is derived. The general solution for the governing equation is expressed by six unknown functions. Secondly, in terms of boundary conditions of the two longitudinal sides of the beam, six functional equations are yielded. These equations are simplified to derive the unified equations to solve the boundary value problems of piezoelectric plane beam. Finally, several examples show the correctness and generalization of this method.

Keywords: Functional equation; Piezoelectric plane beam; Exact solution; Arbitrary load; Elasticity theory

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1. Introduction

Piezoelectric materials have been widely used as actuators and sensors in deformation and vibration control due to the coupling between mechanical and electrical fields. Since its coupling characteristics, it is more complicated for analysis and design of such intelligent structural system as compared with traditional structural system. That is the reason why there are so many numerical models for piezoelectric structure as can be seen in review papers [1-3]. Since the elasticity solutions for simple form of piezoelectric structure can be regarded as the benchmark for verifying the various numerical models, it has also attracted many scientists and engineers to do with this problem.

The necessity to guess and modify the form of stress function and electric displacement function to obtain the elasticity solutions for piezoelectric plane beam subjected to various simple form of loads can be seen in almost each paper that deal with these thesis or related topics. We only list the paper that published in recent years, such as stress function and electric displacement function of Eqs. (14) and (15) in Ref. [4], Eq. (16) in Ref. [5], Eq. (17) in Ref. [6], Eq. (16) in Ref. [7], Eqs. (5) and (6) in Ref. [8], Eq. (5) in Ref. [9], Eq. (9) in Ref. [10], Eq. (6) in Ref. [11], Eq. (15) in Ref. [12], Eq. (6) in Ref. [13]. Usually, they all need to guess the form of stress function and electric displacement function before carry out their solution procedure and can only deal with one specific problem. If the boundary conditions are changed, whether boundary conditions of the two longitudinal sides or boundary conditions of the ends supported conditions, the previous assumptions can not be used. It is also the reason that most of the model given in these papers are simply supported beam or cantilever beam. For other type of ends supported conditions, such as Fixed end – Fixed end piezoelectric plane beam, there are little report about it. Huang [4] only gives a method of how to solve the problem when the piezoelectric beam acted by pressure load that can be transferred into sinusoidal series and need the value of function which represent the load must be equated to zero at the two ends of the beam. The assumption about load in Ref. [4] does not accord with practical conditions. Moreover, the hypothesis of stress function and electrical displacement function in Ref. [4] are not suite for other type of loads, such as shear force or electrical loads.

This paper considers the behavior of piezoelectric plane beam subjected to arbitrary mechanical and electrical loads. Comparing this general method with the method given in the paper that list in Refs. [4-13], the most advantage is it can obtain the solutions directly and does not need to guess and modify the form of stress function or electrical displacement function. Furthermore, it can deal with arbitrary mechanical

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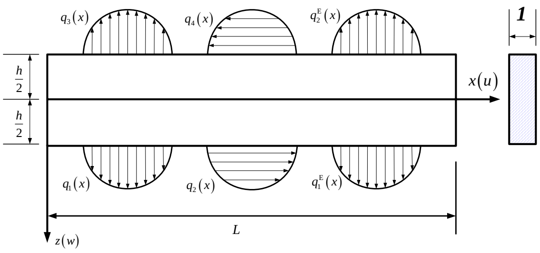


Fig. 1. Piezoelectric plane beam subjected to arbitrary mechanical and electrical loads.

and electrical loads, which can not be realized in any open literature to the best knowledge of the authors.

2. Theoretical formulations

Consider a piezoelectric plane beam with rectangular cross section subjected to arbitrary loads as shown in Fig. 1. Suppose the width of beam is unit, and the length and height are, respectively, L and h.

In $x-z$ plane, the constitutive equations for piezoelectric material can be expressed as Ref. [14]

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \varepsilon_{xz}\end{Bmatrix}=\begin{bmatrix}s_{11} & s_{13} & 0\\ s_{13} & s_{33} & 0\\ 0 & 0 & s_{44}\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{Bmatrix}+\begin{bmatrix}0 & d_{31}\\ 0 & d_{33}\\ d_{15} & 0\end{bmatrix}\begin{Bmatrix}E_x\\ E_z\end{Bmatrix}$$
$$\begin{Bmatrix}D_x\\ D_z\end{Bmatrix}=\begin{bmatrix}0 & 0 & d_{15}\\ d_{31} & d_{33} & 0\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{Bmatrix}+\begin{bmatrix}\delta_{11} & 0\\ 0 & \delta_{33}\end{bmatrix}\begin{Bmatrix}E_x\\ E_z\end{Bmatrix} \tag{1}$$

where σ_{xx}, σ_{zz}, σ_{xz} denote the stress components, ε_{xx}, ε_{zz}, ε_{xz} are the strain components, u and w displacement components, D_x and D_z electric displacement components, E_x and E_z electric field components. s_{11}, s_{13}, s_{33} and s_{44} denote coefficients of elastic compliance. d_{31}, d_{33} and d_{15} are piezoelectricity coefficients. δ_{11} and δ_{33} are dielectric impermeability coefficients.

Considering the following boundary conditions of the two longitudinal sides of the beam

$$\sigma_{zz}\left(x,+\frac{h}{2}\right)=q_1(x),\quad \sigma_{zz}\left(x,-\frac{h}{2}\right)=q_3(x)$$
$$\sigma_{xz}\left(x,+\frac{h}{2}\right)=q_2(x),\quad \sigma_{xz}\left(x,-\frac{h}{2}\right)=q_4(x) \tag{2}$$
$$D_z\left(x,+\frac{h}{2}\right)=q_1^{\mathrm{E}}(x),\quad D_z\left(x,-\frac{h}{2}\right)=q_2^{\mathrm{E}}(x)$$

where $q_i(x)(i=1,2,3,4)$ and $q_i^{\mathrm{E}}(x)(i=1,2)$ represent the arbitrary mechanical and electrical loads, respectively.

The kinematic relation are

$$\varepsilon_{xx}=\frac{\partial u}{\partial x},\quad \varepsilon_{zz}=\frac{\partial w}{\partial z},\quad \varepsilon_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$
$$E_x=-\frac{\partial \Phi_{\mathrm{E}}}{\partial x},\quad E_z=-\frac{\partial \Phi_{\mathrm{E}}}{\partial z} \tag{3}$$

where Φ_{E} represent electrical potential.

The equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xz}}{\partial z}=0,\quad \frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z}=0$$
$$\frac{\partial D_x}{\partial x}+\frac{\partial D_z}{\partial z}=0. \tag{4}$$

From Eq. $(3)_{1\text{-}3}$ the following compatibility equation is obtained

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2}+\frac{\partial^2 \varepsilon_{zz}}{\partial x^2}=\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}. \tag{5}$$

By virtue of Eq. (1), obtain

$$\varepsilon_{xx}=s_{11}\sigma_{xx}+s_{13}\sigma_{zz}-d_{31}\frac{\partial \Phi_{\mathrm{E}}}{\partial z}$$
$$\varepsilon_{zz}=s_{13}\sigma_{xx}+s_{33}\sigma_{zz}-d_{33}\frac{\partial \Phi_{\mathrm{E}}}{\partial z},\quad \varepsilon_{xz}=s_{44}\sigma_{xz}-d_{15}\frac{\partial \Phi_{\mathrm{E}}}{\partial x} \tag{6}$$

$$D_x=d_{15}\sigma_{xz}-\delta_{11}\frac{\partial \Phi_{\mathrm{E}}}{\partial x},\quad D_z=d_{31}\sigma_{xx}+d_{33}\sigma_{zz}-\delta_{33}\frac{\partial \Phi_{\mathrm{E}}}{\partial z}. \tag{7}$$

The stress components can be expressed by using stress function $U_1(x,z)$ as

$$\sigma_{xx}=\frac{\partial^2 U_1}{\partial z^2},\quad \sigma_{zz}=\frac{\partial^2 U_1}{\partial x^2},\quad \sigma_{xz}=-\frac{\partial^2 U_1}{\partial x \partial z}. \tag{8}$$

Substituting Eq. (8) into Eq. (6) and applying Eq. (5) obtains

$$s_{11}\frac{\partial^4 U_1}{\partial z^4}+\left(2s_{13}+s_{44}\right)\frac{\partial^4 U_1}{\partial x^2 \partial z^2}+s_{33}\frac{\partial^4 U_1}{\partial x^4}$$
$$-d_{31}\frac{\partial^3 \Phi_{\mathrm{E}}}{\partial z^3}+\left(d_{15}-d_{33}\right)\frac{\partial^3 \Phi_{\mathrm{E}}}{\partial x^2 \partial z}=0. \tag{9}$$

Substituting Eq. (8) into Eq. (7) and applying Eq. $(4)_3$ obtain

$$d_{31}\frac{\partial^3 U_1}{\partial z^3}-\left(d_{15}-d_{33}\right)\frac{\partial^3 U_1}{\partial x^2 \partial z}-\delta_{33}\frac{\partial^2 \Phi_{\mathrm{E}}}{\partial z^2}-\delta_{11}\frac{\partial^2 \Phi_{\mathrm{E}}}{\partial x^2}=0. \tag{10}$$

Differentiating Eq. (10) with respect to z once and combining Eq. (9) to eliminate $\partial^3 \Phi_{\mathrm{E}} / (\partial x^2 \partial z)$ obtains

$$\frac{\partial^3 \Phi_{\mathrm{E}}}{\partial z^3}=a_{21}\frac{\partial^4 U_1}{\partial z^4}+a_{22}\frac{\partial^4 U_1}{\partial x^2 \partial z^2}+a_{23}\frac{\partial^4 U_1}{\partial x^4}. \tag{11}$$

In this section, the concrete expressions of a_{ij} are given in Appendix A. Letting

$$U_1(x,z)=\frac{\partial U_2(x,z)}{\partial z}. \tag{12}$$

Substituting Eq. (12) into Eqs. (9) and (10) obtains

$$s_{11}\frac{\partial^5 U_2}{\partial z^5}+\left(2s_{13}+s_{44}\right)\frac{\partial^5 U_2}{\partial x^2\partial z^3}+s_{33}\frac{\partial^5 U_2}{\partial x^4\partial z} -d_{31}\frac{\partial^3 \Phi_E}{\partial z^3}+\left(d_{15}-d_{33}\right)\frac{\partial^3 \Phi_E}{\partial x^2\partial z}=0 \quad (13)$$

$$d_{31}\frac{\partial^4 U_2}{\partial z^4}-\left(d_{15}-d_{33}\right)\frac{\partial^4 U_2}{\partial x^2\partial z^2} -\delta_{33}\frac{\partial^2 \Phi_E}{\partial z^2}-\delta_{11}\frac{\partial^2 \Phi_E}{\partial x^2}=0. \quad (14)$$

Integrating Eq. (13) with respect to z once obtains

$$s_{11}\frac{\partial^4 U_2}{\partial z^4}+\left(2s_{13}+s_{44}\right)\frac{\partial^4 U_2}{\partial x^2\partial z^2}+s_{33}\frac{\partial^4 U_2}{\partial x^4} -d_{31}\frac{\partial^2 \Phi_E}{\partial z^2}+\left(d_{15}-d_{33}\right)\frac{\partial^2 \Phi_E}{\partial x^2}=f_1^{(4)}(x) \quad (15)$$

where $f_1(x)$ are arbitrary differentiable function, $f_j^{(i)}(x)$ denote the *i*-th derivative of $f_j(x)$.

Combining Eqs. (14) and (15) obtains

$$\frac{\partial^2 \Phi_E}{\partial x^2}=a_{11}\frac{\partial^4 U_2}{\partial z^4}+a_{12}\frac{\partial^4 U_2}{\partial x^2\partial z^2}+a_{13}\frac{\partial^4 U_2}{\partial x^4}+a_{14}f_1^{(4)}(x) \quad (16)$$

$$\frac{\partial^2 \Phi_E}{\partial z^2}=a_{21}\frac{\partial^4 U_2}{\partial z^4}+a_{22}\frac{\partial^4 U_2}{\partial x^2\partial z^2}+a_{23}\frac{\partial^4 U_2}{\partial x^4}+a_{24}f_1^{(4)}(x) \quad (17)$$

Differentiating Eqs. (16) and (17) with respect to z and x twice, respectively, and letting them equate to each other obtains

$$\frac{\partial^6 U_3}{\partial z^6}+\frac{\left(2s_{13}+s_{44}\right)\delta_{33}+s_{11}\delta_{11}+2d_{31}\left(d_{15}-d_{33}\right)}{s_{11}\delta_{33}-d_{31}^2}\frac{\partial^6 U_3}{\partial z^4\partial x^2} +\frac{\left(2s_{13}+s_{44}\right)\delta_{11}+s_{33}\delta_{33}-\left(d_{15}-d_{33}\right)^2}{s_{11}\delta_{33}-d_{31}^2}\frac{\partial^6 U_3}{\partial z^2\partial x^4} +\frac{s_{33}\delta_{11}}{s_{11}\delta_{33}-d_{31}^2}\frac{\partial^6 U_3}{\partial x^6}=0 \quad (18)$$

where $U_3(x,z)=U_2(x,z)-f_1(x)/s_{33}$. $f_1(x)$ can be regarded as a assistant function when we derive Eq. (18) and take no effect hereafter.

Eq. (18) is the governing equation for piezoelectric beam. The characteristic equation of Eq. (18) is

$$\lambda^6-\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\lambda^4+\left(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2\right)\lambda^2-\lambda_1^2\lambda_2^2\lambda_3^2=0 \quad (19)$$

where λ_1, λ_2, λ_3 are characteristic roots and $\lambda_1\neq\lambda_2\neq\lambda_3$.

The relationship between the characteristic roots and material properties can be obtained by comparing Eq. (19) with Eq. (18) of the coefficients of λ as follows:

$$\begin{cases}\lambda_1^2+\lambda_2^2+\lambda_3^2=-\dfrac{\left(2s_{13}+s_{44}\right)\delta_{33}+s_{11}\delta_{11}+2d_{31}\left(d_{15}-d_{33}\right)}{s_{11}\delta_{33}-d_{31}^2}\\ \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2=\dfrac{\left(2s_{13}+s_{44}\right)\delta_{11}+s_{33}\delta_{33}-\left(d_{15}-d_{33}\right)^2}{s_{11}\delta_{33}-d_{31}^2}\\ \lambda_1^2\lambda_2^2\lambda_3^2=-\dfrac{s_{33}\delta_{11}}{s_{11}\delta_{33}-d_{31}^2}\end{cases} \quad (20)$$

The general solution of Eq. (18) is

$$U_3(x,z)=\varphi_1(x+\lambda_1 z)+\varphi_2(x-\lambda_1 z)+\varphi_3(x+\lambda_2 z) +\varphi_4(x-\lambda_2 z)+\varphi_5(x+\lambda_3 z)+\varphi_6(x-\lambda_3 z) \quad (21)$$

where $\varphi_i(\cdot)(i=1,2,...,6)$ are six arbitrary functions.

By using Eqs. (21), (12) and (8) obtain

$$\sigma_{xx}=\lambda_1^3\left[\varphi_1^{(3)}(x+\lambda_1 z)-\varphi_2^{(3)}(x-\lambda_1 z)\right] +\lambda_2^3\left[\varphi_3^{(3)}(x+\lambda_2 z)-\varphi_4^{(3)}(x-\lambda_2 z)\right] +\lambda_3^3\left[\varphi_5^{(3)}(x+\lambda_3 z)-\varphi_6^{(3)}(x-\lambda_3 z)\right] \quad (22)$$

$$\sigma_{zz}=\lambda_1\left[\varphi_1^{(3)}(x+\lambda_1 z)-\varphi_2^{(3)}(x-\lambda_1 z)\right] +\lambda_2\left[\varphi_3^{(3)}(x+\lambda_2 z)-\varphi_4^{(3)}(x-\lambda_2 z)\right] +\lambda_3\left[\varphi_5^{(3)}(x+\lambda_3 z)-\varphi_6^{(3)}(x-\lambda_3 z)\right] \quad (23)$$

$$\sigma_{xz}=-\lambda_1^2\left[\varphi_1^{(3)}(x+\lambda_1 z)+\varphi_2^{(3)}(x-\lambda_1 z)\right] -\lambda_2^2\left[\varphi_3^{(3)}(x+\lambda_2 z)+\varphi_4^{(3)}(x-\lambda_2 z)\right] -\lambda_3^2\left[\varphi_5^{(3)}(x+\lambda_3 z)+\varphi_6^{(3)}(x-\lambda_3 z)\right] \quad (24)$$

where $\varphi_j^{(i)}(\cdot)$ denote the *i*-th derivative of $\varphi_j(\cdot)$.

By applying Eq. (21), Eq. (12) and integrating Eq. (11) with respect to z thrice obtains

$$\Phi_E=a_{71}\left[\varphi_1^{(2)}(x+\lambda_1 z)+\varphi_2^{(2)}(x-\lambda_1 z)\right] +a_{72}\left[\varphi_3^{(2)}(x+\lambda_2 z)+\varphi_4^{(2)}(x-\lambda_2 z)\right] +a_{73}\left[\varphi_5^{(2)}(x+\lambda_3 z)+\varphi_6^{(2)}(x-\lambda_3 z)\right] +f_2(x)\frac{z^2}{2}+f_3(x)z+f_4(x) \quad (25)$$

where $f_2(x)$, $f_3(x)$ and $f_4(x)$ are arbitrary functions.

The displacement and electric displacement components can be yielded by using Eq. $(3)_{1\text{-}2}$, Eqs. (6) and (7) as

$$u=a_{31}\left[\varphi_1^{(2)}(x+\lambda_1 z)-\varphi_2^{(2)}(x-\lambda_1 z)\right] +a_{32}\left[\varphi_3^{(2)}(x+\lambda_2 z)-\varphi_4^{(2)}(x-\lambda_2 z)\right] +a_{33}\left[\varphi_5^{(2)}(x+\lambda_3 z)-\varphi_6^{(2)}(x-\lambda_3 z)\right] -d_{31}z\int f_2(x)dx-d_{31}\int f_3(x)dx+g_1(z) \quad (26)$$

$$\begin{aligned} w = & a_{41}\left[\varphi_1^{(2)}(x+\lambda_1 z)+\varphi_2^{(2)}(x-\lambda_1 z)\right] \\ & + a_{42}\left[\varphi_3^{(2)}(x+\lambda_2 z)+\varphi_4^{(2)}(x-\lambda_2 z)\right] \\ & + a_{43}\left[\varphi_5^{(2)}(x+\lambda_3 z)+\varphi_6^{(2)}(x-\lambda_3 z)\right] \\ & - d_{33} f_2(x)\frac{z^2}{2} - d_{33} f_3(x) z + f_5(x) \end{aligned} \tag{27}$$

$$\begin{aligned} D_x = & a_{51}\left[\varphi_1^{(3)}(x+\lambda_1 z)+\varphi_2^{(3)}(x-\lambda_1 z)\right] \\ & + a_{52}\left[\varphi_3^{(3)}(x+\lambda_2 z)+\varphi_4^{(3)}(x-\lambda_2 z)\right] \\ & + a_{53}\left[\varphi_5^{(3)}(x+\lambda_3 z)+\varphi_6^{(3)}(x-\lambda_3 z)\right] \\ & - \delta_{11} f_2^{(1)}(x)\frac{z^2}{2} - \delta_{11} f_3^{(1)}(x) z - \delta_{11} f_4^{(1)}(x) \end{aligned} \tag{28}$$

$$\begin{aligned} D_z = & a_{61}\left[\varphi_1^{(3)}(x+\lambda_1 z)-\varphi_2^{(3)}(x-\lambda_1 z)\right] \\ & + a_{62}\left[\varphi_3^{(3)}(x+\lambda_2 z)-\varphi_4^{(3)}(x-\lambda_2 z)\right] \\ & + a_{63}\left[\varphi_5^{(3)}(x+\lambda_3 z)-\varphi_6^{(3)}(x-\lambda_3 z)\right] \\ & - \delta_{33} f_2(x) z - \delta_{33} f_3(x) \end{aligned} \tag{29}$$

where $f_5(x)$ and $g_1(z)$ are arbitrary functions.

By using Eqs. $(3)_3$, $(4)_3$ (18) and (26)-(29) obtain

$$\begin{aligned} & (d_{15}-d_{33}) f_2^{(1)}(x)\frac{z^2}{2} + (d_{15}-d_{33}) f_3^{(1)}(x) z + f_5^{(1)}(x) \\ & \quad + d_{15} f_4^{(1)}(x) - d_{31}\int f_2(x)dx + g_1^{(1)}(z) = 0 \end{aligned} \tag{30}$$

$$\delta_{11} f_2^{(2)}(x)\frac{z^2}{2} + \delta_{11} f_3^{(2)}(x) z + \delta_{11} f_4^{(2)}(x) + \delta_{33} f_2(x) = 0. \tag{31}$$

Eq. (31) can be regarded as the quadratic algebra equation with respect to z. Usually, Eq. (31) can only have two roots. However, Eq. (31) must be satisfied for arbitrary value of z in the region $[-h/2, h/2]$. The only possible situation is the coefficients of z with any degree equate to zero, which imply that

$$f_2^{(2)}(x) = 0, \quad f_3^{(2)}(x) = 0, \quad \delta_{11} f_4^{(2)}(x) + \delta_{33} f_2(x) = 0. \tag{32}$$

Integrating Eq. (32) with respect to x twice obtain

$$\begin{aligned} & f_2(x) = C_1 x + C_2, \quad f_3(x) = C_3 x + C_4 \\ & f_4(x) = -\frac{\delta_{33}}{\delta_{11}}\left(C_1\frac{x^3}{6} + C_2\frac{x^2}{2}\right) + C_7 x + C_8. \end{aligned} \tag{33}$$

Substituting Eq. (33) into Eq. (30) obtain

$$\begin{aligned} & g_1^{(1)}(z) + (d_{15}-d_{33})\left(C_1\frac{z^2}{2} + C_3 z\right) + f_5^{(1)}(x) \\ & -\left(d_{31} + d_{15}\frac{\delta_{33}}{\delta_{11}}\right)\left(C_1\frac{x^2}{2} + C_2 x\right) - d_{31}C_5 + d_{15}C_7 = 0. \end{aligned} \tag{34}$$

Since $f_5(x)$ and $g_1(z)$ are the single variable functions of x and z, respectively, which show that

$$\begin{aligned} f_5(x) = & \left(d_{31} + d_{15}\frac{\delta_{33}}{\delta_{11}}\right)\left(C_1\frac{x^3}{6} + C_3\frac{x^2}{2}\right) \\ & + (d_{31}C_5 - d_{15}C_7 + C_6) x + C_{10} \end{aligned} \tag{35}$$

$$g_1(z) = -(d_{15}-d_{33})\left(C_1\frac{z^3}{6} + C_3\frac{z^2}{2}\right) + C_6 z + C_9 \tag{36}$$

where $C_i\,(i = 1,2,\ldots,10)$ are unknown integral constants.

Solutions must satisfy the boundary conditions at the two longitudinal sides of the beam exactly. By means of Eq. (2), Eqs. (23), (24) and (29) yield six functional equations

$$\begin{aligned} & \lambda_1\left[\varphi_1^{(3)}\left(x+\lambda_1\frac{h}{2}\right) - \varphi_2^{(3)}\left(x-\lambda_1\frac{h}{2}\right)\right] \\ & +\lambda_2\left[\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right) - \varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\right] \\ & +\lambda_3\left[\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right) - \varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\right] = q_1(x) \end{aligned} \tag{37}$$

$$\begin{aligned} & \lambda_1^2\left[\varphi_1^{(3)}\left(x+\lambda_1\frac{h}{2}\right) + \varphi_2^{(3)}\left(x-\lambda_1\frac{h}{2}\right)\right] \\ & +\lambda_2^2\left[\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right) + \varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\right] \\ & +\lambda_3^2\left[\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right) + \varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\right] = -q_2(x) \end{aligned} \tag{38}$$

$$\begin{aligned} & \lambda_1\left[\varphi_1^{(3)}\left(x-\lambda_1\frac{h}{2}\right) - \varphi_2^{(3)}\left(x+\lambda_1\frac{h}{2}\right)\right] \\ & +\lambda_2\left[\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right) - \varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\right] \\ & +\lambda_3\left[\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right) - \varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\right] = q_3(x) \end{aligned} \tag{39}$$

$$\begin{aligned} & \lambda_1^2\left[\varphi_1^{(3)}\left(x-\lambda_1\frac{h}{2}\right) + \varphi_2^{(3)}\left(x+\lambda_1\frac{h}{2}\right)\right] \\ & +\lambda_2^2\left[\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right) + \varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\right] \\ & +\lambda_3^2\left[\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right) + \varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\right] = -q_4(x) \end{aligned} \tag{40}$$

$$\begin{aligned} & a_{61}\left[\varphi_1^{(3)}\left(x+\lambda_1\frac{h}{2}\right) - \varphi_2^{(3)}\left(x-\lambda_1\frac{h}{2}\right)\right] \\ & +a_{62}\left[\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right) - \varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\right] \\ & +a_{63}\left[\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right) - \varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\right] = q_{1\mathrm{C}}^{\mathrm{E}}(x) \end{aligned} \tag{41}$$

$$\begin{aligned} & a_{61}\left[\varphi_1^{(3)}\left(x-\lambda_1\frac{h}{2}\right) - \varphi_2^{(3)}\left(x+\lambda_1\frac{h}{2}\right)\right] \\ & +a_{62}\left[\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right) - \varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\right] \\ & +a_{63}\left[\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right) - \varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\right] = q_{2\mathrm{C}}^{\mathrm{E}}(x) \end{aligned} \tag{42}$$

where

$$q_{1C}^{E}(x) = q_1^{E}(x) + \left[\delta_{33}(C_1 x + C_2)h\right]/2 + \delta_{33}(C_3 x + C_4),$$
$$q_{2C}^{E}(x) = q_2^{E}(x) - \left[\delta_{33}(C_1 x + C_2)h\right]/2 + \delta_{33}(C_3 x + C_4).$$

Multiplying Eq. (37) by λ_1 then minus Eq. (38) obtains

$$\begin{aligned}&\varphi_2^{(3)}\left(x-\lambda_1\frac{h}{2}\right)-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)=-\frac{\lambda_1 q_1(x)+q_2(x)}{2\lambda_1^2}.\end{aligned}\tag{43}$$

Multiplying Eq. (39) by λ_1 then minus Eq. (40) obtains

$$\begin{aligned}&\varphi_2^{(3)}\left(x+\lambda_1\frac{h}{2}\right)-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)=-\frac{\lambda_1 q_3(x)+q_4(x)}{2\lambda_1^2}.\end{aligned}\tag{44}$$

Replacing x with $x+\lambda_1 h/2$ in Eq. (43) and x with $x-\lambda_1 h/2$ in Eq. (44), respectively, obtains

$$\begin{aligned}&\varphi_2^{(3)}(x)-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&=-\frac{\lambda_1 q_1(x+\lambda_1 h/2)+q_2(x+\lambda_1 h/2)}{2\lambda_1^2}\end{aligned}\tag{45}$$

$$\begin{aligned}&\varphi_2^{(3)}(x)-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&=-\frac{\lambda_1 q_3(x-\lambda_1 h/2)+q_4(x-\lambda_1 h/2)}{2\lambda_1^2}.\end{aligned}\tag{46}$$

Eq. (45) minus Eq. (46) obtains

$$\begin{aligned}&\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\left[\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)+\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\right]\\&+\frac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&=\frac{\lambda_1 q_1(x+\lambda_1 h/2)+q_2(x+\lambda_1 h/2)}{\lambda_2(\lambda_1-\lambda_2)}\\&-\frac{\lambda_1 q_3(x-\lambda_1 h/2)+q_4(x-\lambda_1 h/2)}{\lambda_2(\lambda_1-\lambda_2)}.\end{aligned}\tag{47}$$

Multiplying Eq. (37) by λ_1 then plus Eq. (38) obtain

$$\begin{aligned}&\varphi_1^{(3)}\left(x+\lambda_1\frac{h}{2}\right)+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_1(x)-q_2(x)}{2\lambda_1^2}.\end{aligned}\tag{48}$$

Multiplying Eq. (40) by λ_1 then plus Eq. (41) obtain

$$\begin{aligned}&\varphi_1^{(3)}\left(x-\lambda_1\frac{h}{2}\right)+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_3(x)-q_4(x)}{2\lambda_1^2}.\end{aligned}\tag{49}$$

Replacing x with $x-\lambda_1 h/2$ in Eq. (48) and x with $x+\lambda_1 h/2$ in Eq. (49), respectively, obtain

$$\begin{aligned}&\varphi_1^{(3)}(x)+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&=\frac{\lambda_1 q_1(x-\lambda_1 h/2)-q_2(x-\lambda_1 h/2)}{2\lambda_1^2}\end{aligned}\tag{50}$$

$$
\begin{aligned}
&\varphi_1^{(3)}(x)+\frac{\lambda_1\lambda_2+\lambda_2^2}{2\lambda_1^2}\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\
&\quad-\frac{\lambda_1\lambda_2-\lambda_2^2}{2\lambda_1^2}\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\
&\quad+\frac{\lambda_1\lambda_3+\lambda_3^2}{2\lambda_1^2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{\lambda_1\lambda_3-\lambda_3^2}{2\lambda_1^2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad=\frac{\lambda_1 q_3(x+\lambda_1 h/2)-q_4(x+\lambda_1 h/2)}{2\lambda_1^2}.
\end{aligned} \tag{51}
$$

Eq. (50) minus Eq. (51) obtains

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\
&-\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\left[\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\right]\\
&\quad+\frac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad+\frac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad=\frac{\lambda_1 q_1(x-\lambda_1 h/2)+q_4(x+\lambda_1 h/2)}{\lambda_2(\lambda_1+\lambda_2)}\\
&\quad-\frac{\lambda_1 q_3(x+\lambda_1 h/2)+q_2(x-\lambda_1 h/2)}{\lambda_2(\lambda_1+\lambda_2)}.
\end{aligned} \tag{52}
$$

Multiplying Eq. (37) by a_{61}/λ_1 then minus Eq. (41) obtains

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x+\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_2\frac{h}{2}\right)\\
&+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\left[\varphi_5^{(3)}\left(x+\lambda_3\frac{h}{2}\right)-\varphi_6^{(3)}\left(x-\lambda_3\frac{h}{2}\right)\right]\\
&\quad=\frac{\lambda_1 q_{1C}^{E}(x)-a_{61}q_1(x)}{a_{62}\lambda_1-a_{61}\lambda_2}.
\end{aligned} \tag{53}
$$

Multiplying Eq. (39) by a_{61}/λ_1 then minus Eq. (42) obtains

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x+\lambda_2\frac{h}{2}\right)\\
&+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\left[\varphi_5^{(3)}\left(x-\lambda_3\frac{h}{2}\right)-\varphi_6^{(3)}\left(x+\lambda_3\frac{h}{2}\right)\right]\\
&\quad=\frac{\lambda_1 q_{2C}^{E}(x)-a_{61}q_3(x)}{a_{62}\lambda_1-a_{61}\lambda_2}.
\end{aligned} \tag{54}
$$

Replacing x with $x+\lambda_1 h/2$ in Eq. (53) and x with $x-\lambda_1 h/2$ in Eq. (54), respectively, obtain

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\
&\quad+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad=\frac{\lambda_1 q_{1C}^{E}(x+\lambda_1 h/2)-a_{61}q_1(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\
&\quad+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad=\frac{\lambda_1 q_{2C}^{E}(x-\lambda_1 h/2)-a_{61}q_3(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}.
\end{aligned} \tag{56}
$$

Eq. (55) minus Eq. (56) obtains

$$
\begin{aligned}
&\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\
&-\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)+\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\
&\quad+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\
&\quad+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\
&\quad=\frac{a_{61}q_3(x-\lambda_1 h/2)+\lambda_1 q_{1C}^{E}(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}\\
&\quad-\frac{a_{61}q_1(x+\lambda_1 h/2)+\lambda_1 q_{2C}^{E}(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}.
\end{aligned} \tag{57}
$$

Eq. (57) minus Eq. (47) obtains

$$
\begin{aligned}
&a_{86}^{1}\left[\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\right]\\
&+a_{83}^{1}\left[\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)-\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\right]\\
&-a_{84}^{1}\left[\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)-\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\right]=k_{11}(x)
\end{aligned} \tag{58}
$$

where

$$k_{11}(x)=\frac{a_{61}q_3(x-\lambda_1 h/2)+\lambda_1 q_{1C}^{E}(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{a_{61}q_1(x+\lambda_1 h/2)+\lambda_1 q_{2C}^{E}(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{\lambda_1 q_1(x+\lambda_1 h/2)+q_2(x+\lambda_1 h/2)}{\lambda_2(\lambda_1-\lambda_2)}+\frac{\lambda_1 q_3(x-\lambda_1 h/2)+q_4(x-\lambda_1 h/2)}{\lambda_2(\lambda_1-\lambda_2)}. \quad (59)$$

Replacing x with $x-\lambda_1 h/2$ in Eq. (53) and x with $x+\lambda_1 h/2$ in Eq. (54), respectively, obtain

$$\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_{1C}^{E}(x-\lambda_1 h/2)-a_{61}q_1(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2} \quad (60)$$

$$\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_{2C}^{E}(x+\lambda_1 h/2)-a_{61}q_3(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}. \quad (61)$$

Eq. (60) minus Eq. (61) obtains

$$\begin{aligned}&\varphi_3^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_3^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)+\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&=\frac{a_{61}q_3(x+\lambda_1 h/2)+\lambda_1 q_{1C}^{E}(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{a_{61}q_1(x-\lambda_1 h/2)+\lambda_1 q_{2C}^{E}(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}.\end{aligned} \quad (62)$$

Eq. (62) minus Eq. (52) obtains

$$a_{85}^{1}\left[\varphi_4^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)-\varphi_4^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\right]+a_{81}^{1}\left[\varphi_5^{(3)}\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)-\varphi_5^{(3)}\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\right]-a_{82}^{1}\left[\varphi_6^{(3)}\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)-\varphi_6^{(3)}\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\right]=k_{12}(x) \quad (63)$$

where

$$k_{12}(x)=\frac{a_{61}q_3(x+\lambda_1 h/2)+\lambda_1 q_{1C}^{E}(x-\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{a_{61}q_1(x-\lambda_1 h/2)+\lambda_1 q_{2C}^{E}(x+\lambda_1 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{\lambda_1 q_1(x-\lambda_1 h/2)+q_4(x+\lambda_1 h/2)}{\lambda_2(\lambda_1+\lambda_2)}+\frac{\lambda_1 q_3(x+\lambda_1 h/2)+q_2(x-\lambda_1 h/2)}{\lambda_2(\lambda_1+\lambda_2)}. \quad (64)$$

Replacing x with $x-\lambda_2 h/2$ in Eq. (53) and x with $x+\lambda_2 h/2$ in Eq. (54), respectively, obtain

$$\varphi_3^{(3)}(x)-\varphi_4^{(3)}(x-\lambda_2 h)+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x-\lambda_2\frac{h}{2}+\lambda_3\frac{h}{2}\right)-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x-\lambda_2\frac{h}{2}-\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_{1C}^{E}(x-\lambda_2 h/2)-a_{61}q_1(x-\lambda_2 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2} \quad (65)$$

$$\varphi_3^{(3)}(x)-\varphi_4^{(3)}(x+\lambda_2 h)+\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_5^{(3)}\left(x+\lambda_2\frac{h}{2}-\lambda_3\frac{h}{2}\right)-\frac{a_{63}\lambda_1-a_{61}\lambda_3}{a_{62}\lambda_1-a_{61}\lambda_2}\varphi_6^{(3)}\left(x+\lambda_2\frac{h}{2}+\lambda_3\frac{h}{2}\right)=\frac{\lambda_1 q_{2C}^{E}(x+\lambda_2 h/2)-a_{61}q_3(x+\lambda_2 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}. \quad (66)$$

Eq. (65) minus Eq. (66) obtains

$$\left[\varphi_4^{(3)}(x+\lambda_2 h)-\varphi_4^{(3)}(x-\lambda_2 h)\right]+a_{80}^{1}\left[\varphi_5^{(3)}\left(x-\lambda_2\frac{h}{2}+\lambda_3\frac{h}{2}\right)+\varphi_6^{(3)}\left(x+\lambda_2\frac{h}{2}+\lambda_3\frac{h}{2}\right)\right]-a_{80}^{1}\left[\varphi_5^{(3)}\left(x+\lambda_2\frac{h}{2}-\lambda_3\frac{h}{2}\right)+\varphi_6^{(3)}\left(x-\lambda_2\frac{h}{2}-\lambda_3\frac{h}{2}\right)\right]=k_{13}(x) \quad (67)$$

where

$$k_{13}(x)=\frac{a_{61}q_3(x+\lambda_2 h/2)+\lambda_1 q_{1C}^{E}(x-\lambda_2 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}-\frac{a_{61}q_1(x-\lambda_2 h/2)-\lambda_1 q_{2C}^{E}(x+\lambda_2 h/2)}{a_{62}\lambda_1-a_{61}\lambda_2}. \tag{68}$$

Eqs. (58), (63) and (67) are the unified equations to solve the boundary value problem of piezoelectric plane beam acted by arbitrary mechanical and electrical loads.

We can construct $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ in terms of $k_{1i}(x)\,(i=1,2,3)$. Once $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ are constructed, $\varphi_3(x)$ can be constructed by replacing x with $x-\lambda_2 h/2$ in Eq. (53) and integrating Eq. (53) with respect to x thrice

$$\begin{aligned}\varphi_3(x)=&\varphi_4(x-\lambda_2 h)-a_{80}^{1}\varphi_5\left(x-\lambda_2\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&+a_{80}^{1}\varphi_6\left(x-\lambda_2\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_1}{a_{62}\lambda_1-a_{61}\lambda_2}\int\left\{\int\left[\int q_{1C}^{E}\left(x-\lambda_2\frac{h}{2}\right)dx\right]dx\right\}dx\\&-\frac{a_{61}}{a_{62}\lambda_1-a_{61}\lambda_2}\int\left\{\int\left[\int q_1\left(x-\lambda_2\frac{h}{2}\right)dx\right]dx\right\}dx\\&+A_{32}\frac{x^2}{2}+A_{31}x+A_{30}.\end{aligned} \tag{69}$$

$\varphi_2(x)$ can be constructed by integrating Eq. (45) with respect to x thrice

$$\begin{aligned}\varphi_2(x)=&+\frac{\lambda_2(\lambda_1-\lambda_2)}{2\lambda_1^2}\varphi_3\left(x+\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_2(\lambda_1+\lambda_2)}{2\lambda_1^2}\varphi_4\left(x+\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_3(\lambda_1-\lambda_3)}{2\lambda_1^2}\varphi_5\left(x+\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&-\frac{\lambda_3(\lambda_1+\lambda_3)}{2\lambda_1^2}\varphi_6\left(x+\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&-\frac{1}{2\lambda_1}\int\left\{\int\left\{\int\left[q_1\left(x+\lambda_1\frac{h}{2}\right)\right]dx\right\}dx\right\}dx\\&+\frac{1}{2\lambda_1^2}\int\left\{\int\left\{\int\left[q_2\left(x+\lambda_1\frac{h}{2}\right)\right]dx\right\}dx\right\}dx\\&+A_{22}\frac{x^2}{2}+A_{21}x+A_{20}.\end{aligned} \tag{70}$$

$\varphi_1(x)$ can be constructed by integrating Eq. (50) with respect to x thrice

$$\begin{aligned}\varphi_1(x)=&-\frac{\lambda_2(\lambda_1+\lambda_2)}{2\lambda_1^2}\varphi_3\left(x-\lambda_1\frac{h}{2}+\lambda_2\frac{h}{2}\right)\\&+\frac{\lambda_2(\lambda_1-\lambda_2)}{2\lambda_1^2}\varphi_4\left(x-\lambda_1\frac{h}{2}-\lambda_2\frac{h}{2}\right)\\&-\frac{\lambda_3(\lambda_1+\lambda_3)}{2\lambda_1^2}\varphi_5\left(x-\lambda_1\frac{h}{2}+\lambda_3\frac{h}{2}\right)\\&+\frac{\lambda_3(\lambda_1-\lambda_3)}{2\lambda_1^2}\varphi_6\left(x-\lambda_1\frac{h}{2}-\lambda_3\frac{h}{2}\right)\\&+\frac{1}{2\lambda_1}\int\left\{\int\left\{\int\left[q_1\left(x-\lambda_1\frac{h}{2}\right)\right]dx\right\}dx\right\}dx\\&-\frac{1}{2\lambda_1^2}\int\left\{\int\left\{\int\left[q_2\left(x-\lambda_1\frac{h}{2}\right)\right]dx\right\}dx\right\}dx\\&+A_{12}\frac{x^2}{2}+A_{11}x+A_{10}\end{aligned} \tag{71}$$

where A_{ij} are unknown constants.

The expressions of σ_{xx}, σ_{zz}, σ_{xz}, u, w, D_x, D_z and Φ_E can be expressed by using $\varphi_i(x)(i=1,2,...,6)$ at present. Since Eq. (2) are satisfied, the reaming unknown constants can be determined by using the boundary conditions of two ends of the beam and obtain the expressions of stress, displacement, electrical displacement and electrical potential finally.

From the solution procedure mentioned above, how to construct $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ is the key point to figure out the problem. The conclusions given below can be confirmed by substituting the expressions of loads and expressions of $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ into Eqs. (58), (63) and (67).

When the plane beam acted by loads

$$q_j(x)=q_{j0}+\sum_{i=1}^{m}q_{ji}\left(\frac{x}{L}\right)^i \quad (j=1,3) \tag{72}$$

$\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ can be constructed as

$$\varphi_4(x)=\sum_{i=0}^{m+6}A_{4i}x^i,\quad \varphi_5(x)=\sum_{i=0}^{m+6}A_{5i}x^i,\quad \varphi_6(x)=\sum_{i=0}^{m+6}A_{6i}x^i \tag{73}$$

where A_{4i}, A_{5i} and A_{6i} $(i=0,1,...,m+6)$ are unknown constants.

When the plane beam acted by loads

$$q_j(x)=q_{j0}+\sum_{i=1}^{m}q_{ji}\left(\frac{x}{L}\right)^i \quad (j=2,4) \tag{74}$$

$\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ can be constructed as

$$\varphi_4(x)=\sum_{i=0}^{m+5}A_{4i}x^i,\quad \varphi_5(x)=\sum_{i=0}^{m+5}A_{5i}x^i,\quad \varphi_6(x)=\sum_{i=0}^{m+5}A_{6i}x^i \tag{75}$$

where A_{4i}, A_{5i} and A_{6i} $(i=0,1,...,m+5)$ are unknown constants.

When the plane beam acted by loads

$$q_j^{\mathrm{E}}(x) = q_{j0}^{\mathrm{E}} + \sum_{i=1}^{m} q_{ji}\left(\frac{x}{L}\right)^i \quad (j=1,2) \tag{76}$$

$\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ can be constructed as

$$\varphi_4(x) = \sum_{i=0}^{m+4} A_{4i}x^i, \quad \varphi_5(x) = \sum_{i=0}^{m+4} A_{5i}x^i, \quad \varphi_6(x) = \sum_{i=0}^{m+4} A_{6i}x^i \tag{77}$$

where A_{4i}, A_{5i} and A_{6i} $(i = 0,1,...,m+4)$ are unknown constants.

When the plane beam acted by loads

$$q_j(x) = q_j \sin(\delta_0 x) \quad \text{or} \quad q_j(x) = q_j \cos(\delta_0 x) \quad (j=1,3) \tag{78}$$

where δ_0 is a known coefficient, $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ can be constructed as

$$\begin{cases} \varphi_4(x) = \sum_{i=0}^{5} A_{4i}x^i + E_4 \cdot \cos(\delta_0 x) + F_4 \cdot \sin(\delta_0 x) \\ \varphi_5(x) = \sum_{i=0}^{5} A_{5i}x^i + E_5 \cdot \cos(\delta_0 x) + F_5 \cdot \sin(\delta_0 x) \\ \varphi_6(x) = \sum_{i=0}^{5} A_{6i}x^i + E_6 \cdot \cos(\delta_0 x) + F_6 \cdot \sin(\delta_0 x) \end{cases} \tag{79}$$

where A_{4i}, A_{5i} and A_{6i} $(i = 0,1,...,5)$, E_4, F_4, E_5, F_5, E_6 and F_6 are unknown constants.

Similarly, if we need to deal with

$$q_j(x) = q_j \sin(\delta_0 x) \quad \text{or} \quad q_j(x) = q_j \cos(\delta_0 x) \quad (j=2,4) \tag{80}$$

we can simply let the polynomial part of Eq. (79) with degree of 4 against x and 3 for following type of loads.

$$q_j^{\mathrm{E}}(x) = q_j^{\mathrm{E}} \sin(\delta_0 x) \quad \text{or} \quad q_j^{\mathrm{E}}(x) = q_j^{\mathrm{E}} \cos(\delta_0 x) \quad (j=1,2) \tag{81}$$

In the case when the loads are complicated, such as they are not continuous in one of the longitudinal side, we can translate functions, which represent the loads, into Fourier series as

$$q_j(x) = \eta_{j0} + \sum_{n=1}^{\infty}\left[\eta_{jn}\cos(H_n x) + \psi_{jn}\sin(H_n x)\right] \quad (j=1,3) \tag{82}$$

where

$$\begin{cases} \eta_{j0} = \frac{1}{L}\int_0^L q_j(x)dx \\ \eta_{jn} = \frac{2}{L}\int_0^L q_j(x)\cos(H_n x)dx, \quad H_n = \frac{2n\pi}{L} \\ \psi_{jn} = \frac{2}{L}\int_0^L q_j(x)\sin(H_n x)dx \end{cases} \tag{83}$$

$\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ can be constructed as

$$\begin{cases} \varphi_4(x) = \sum_{i=0}^{6} A_{4i}x^i + \sum_{n=1}^{\infty}\left[E_{4n}\cos(H_n x)\right] + \sum_{n=1}^{\infty}\left[F_{4n}\cos(H_n x)\right] \\ \varphi_5(x) = \sum_{i=0}^{6} A_{5i}x^i + \sum_{n=1}^{\infty}\left[E_{5n}\cos(H_n x)\right] + \sum_{n=1}^{\infty}\left[F_{5n}\cos(H_n x)\right] \\ \varphi_6(x) = \sum_{i=0}^{6} A_{6i}x^i + \sum_{n=1}^{\infty}\left[E_{6n}\cos(H_n x)\right] + \sum_{n=1}^{\infty}\left[F_{6n}\cos(H_n x)\right] \end{cases} \tag{84}$$

where A_{4i}, A_{5i} and A_{6i} $(i = 0,1,...,m+6)$, E_{4n}, F_{4n}, E_{5n}, F_{5n}, E_{6n} and F_{6n} are unknown constants.

Analogically, we can deal with discontinuous distribution of $q_j(x)(j=2,4)$ and $q_j^{\mathrm{E}}(x)(j=1,2)$ by only change polynomial part of Eq. (84) with degree of 5 and 4 against x, respectively.

3. Applications

In this section, various boundary conditions, including two longitudinal sides of the beam and two ends supported conditions of the beam, are considered to validating the correctness and generalization of this method. In Section 3.1, the solution procedure based on the idea given in Section 2 is presented in detail. Then the solutions for three additional examples are given. As the loads become complicated, the final expressions become very length, In Sections 3.5, 3.6 and 3.7 the numerical results are given in the form of surface. The material constants are all derived from Ref. [14]. For all the numerical examples, L and h are taken as 0.1 m and 0.02 m, respectively.

3.1 Hinged end–roller end beam subjected to uniform shear force

This example is used to compare the solutions with the results given in Ref. [15]. The boundary conditions of the two longitudinal sides are

$$q_1(x) = 0, \quad q_2(x) = 0, \quad q_3(x) = 0, \quad q_4(x) = q_{40} \\ q_1^{\mathrm{E}}(x) = 0, \quad q_2^{\mathrm{E}}(x) = 0. \tag{85}$$

The boundary conditions of two ends of beam are

$$\begin{gathered} u\big|_{\substack{x=0\\z=0}} = 0, \; w\big|_{\substack{x=0\\z=0}} = 0, \; w\big|_{\substack{x=L\\z=0}} = 0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)dz = 0, \; \Phi_{\mathrm{E}}\big|_{\substack{x=0\\z=0}} = 0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=0}\right)zdz = 0, \quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)zdz = 0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=0}\right)dz = 0, \quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=L}\right)dz = 0. \end{gathered} \tag{86}$$

Step 1: Constructing the form of $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ in terms of the type of load. In this case, Eq. (75) is used and $\varphi_4(x)$, $\varphi_5(x)$ and $\varphi_6(x)$ are constructed as

$$\varphi_4(x) = \sum_{i=0}^{5} A_{4i}x^i, \quad \varphi_5(x) = \sum_{i=0}^{5} A_{5i}x^i, \quad \varphi_6(x) = \sum_{i=0}^{5} A_{6i}x^i. \tag{87}$$

Step 2: Substituting Eq. (87) into the unified equations of Eqs. (58), (63) and (67) obtain three linear algebra equations with respect to x. For the same reason mentioned below Eq. (31), the coefficients of x with any degree must be equated to zero to satisfy the unified equations. It means that the unified equations will provide 6 linear algebra equations with respect to unknown constants A_{44}, A_{54}, A_{64}, A_{45}, A_{55} and A_{65}, which will show us the relationship between (A_{45}, A_{55} and A_{65}) and (A_{44}, A_{54}, A_{64}) in this case.

Step 3: Substituting Eq. (87) into Eqs. (69)-(71) to obtain the explicit expressions of $\varphi_1(x)$, $\varphi_2(x)$ and $\varphi_3(x)$. It is noted that at present the expressions of $\varphi_i(x)(i=1,2,...,6)$ are all explicit polynomial expressions.

Step 4: Substituting $\varphi_i(x)(i=1,2,...,6)$ into Eqs. (22)-(29) and combining Eqs. (32), (35)-(36) obtain the explicit expressions of σ_{xx}, σ_{zz}, σ_{xz}, u, w, D_x, D_z and Φ_{E} with some unknown constants. We can verify that boundary conditions of the two longitudinal sides of the beam have been satisfied.

Step 5: The remaining unknown constants can be determined by using the two ends supported conditions of the beam and obtain the final solutions.

By using the solution procedure given above, the expressions for stress, displacement, electrical displacement and electrical potential are

$$\sigma_{xx}=-\frac{q_{40}(L-x)}{h},\quad \sigma_{zz}=0,\quad \sigma_{xz}=\frac{q_{40}(h-2z)}{2h} \tag{88}$$

$$D_z=0,\quad D_x=-\frac{q_{40}(d_{31}\delta_{11}+d_{15}\delta_{33})z}{\delta_{33}h} \tag{89}$$

$$\Phi_{\mathrm{E}}=-\frac{q_{40}(2d_{31}\delta_{11}Lz-2d_{31}\delta_{11}xz-d_{15}\delta_{33}hx)}{2\delta_{11}\delta_{33}h} \tag{90}$$

$$w=-\frac{q_{40}(s_{13}\delta_{33}-d_{31}d_{33})(L-x)z}{\delta_{33}h} \tag{91}$$

$$\begin{aligned} u=&-\frac{q_{40}(s_{11}\delta_{33}-d_{31}^2)(2L-x)x}{2\delta_{33}h}+\frac{q_{40}s_{44}(h-z)z}{2h}\\ &-\frac{q_{40}\left[s_{13}\delta_{11}\delta_{33}z+\delta_{33}d_{15}^2h+\delta_{11}d_{31}(d_{15}-d_{33})z\right]z}{2\delta_{11}\delta_{33}h}. \end{aligned} \tag{92}$$

We can confirm that Eqs. (88)-(92) satisfy Eqs. (85) and (86). Letting the coefficients of piezoelectricity and dielectric impermeability equate to zero, the solution degenerate to the Hinged end–Roller end orthotropic plane beam subjected to uniform shear force as can be seen in Ref. [15], which show the correctness of this method.

3.2 Hinged end–roller end beam subjected to uniform electric displacement

The boundary conditions of the two longitudinal sides are

$$\begin{gathered} q_1(x)=0,\quad q_2(x)=0,\quad q_3(x)=0,\quad q_4(x)=0\\ q_1^{\mathrm{E}}(x)=0,\quad q_2^{\mathrm{E}}(x)=q_{20}^{\mathrm{E}}. \end{gathered} \tag{93}$$

The ends supported conditions are the same to Eq. (86). By using the solution method given above, the solutions are

$$\sigma_{xx}=0,\quad \sigma_{zz}=0,\quad \sigma_{xz}=0 \tag{94}$$

$$D_z=\frac{q_{20}^{\mathrm{E}}(h-2z)}{2h},\quad D_x=\frac{q_{20}^{\mathrm{E}}(L-x)x}{Lh} \tag{95}$$

$$\Phi_{\mathrm{E}}=-\frac{q_{20}^{\mathrm{E}}\left[\delta_{33}(3L-2x)x^2+3\delta_{11}(h-z)Lz\right]}{6\delta_{11}\delta_{33}Lh} \tag{96}$$

$$\begin{aligned} w=&-\frac{q_{20}^{\mathrm{E}}d_{33}x^3}{3\delta_{11}Lh}+\frac{q_{20}^{\mathrm{E}}(\delta_{11}d_{31}+\delta_{33}d_{15})x^2}{2\delta_{11}\delta_{33}h}\\ &-\frac{q_{20}^{\mathrm{E}}(3\delta_{11}d_{31}+3\delta_{33}d_{15}-2\delta_{33}d_{33})Lx}{6\delta_{11}\delta_{33}h}+\frac{q_{20}^{\mathrm{E}}d_{33}(h-z)z}{2\delta_{33}h}. \end{aligned} \tag{97}$$

We can confirm that Eqs. (94)-(97) satisfy Eqs. (93) and (86). The expression for u are very length and will not list here for concise. It is noted that when piezoelectric beam acted by uniform electrical displacement the components of w arise but the components of stress equate to zero, which can be utilized for deformation control.

3.3 Fixed end–free end beam subjected to uniform shear force

The boundary conditions of the two longitudinal sides are

$$\begin{gathered} q_1(x)=0,\quad q_2(x)=0,\quad q_3(x)=0,\quad q_4(x)=q_{40}\\ q_1^{\mathrm{E}}(x)=0,\quad q_2^{\mathrm{E}}(x)=0. \end{gathered} \tag{98}$$

The boundary conditions of two ends of beam are

$$\begin{gathered} u\big|_{\substack{x=0\\z=0}}=0,\ w\big|_{\substack{x=0\\z=0}}=0,\ \frac{\mathrm{d}w}{\mathrm{d}x}\bigg|_{\substack{x=0\\z=0}}=0,\ \Phi_{\mathrm{E}}\big|_{\substack{x=0\\z=0}}=0\\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xz}\big|_{x=0}\right)dz=0,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=0}\right)dz=0\\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)zdz=0,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=0}\right)dz=0. \end{gathered} \tag{99}$$

The solutions are

$$\begin{gathered} \sigma_{xx}=-\frac{q_{40}(h-6z)(L-x)}{h^2},\quad \sigma_{zz}=0\\ \sigma_{xz}=-\frac{q_{40}(h+6z)(h-2z)}{4h^2} \end{gathered} \tag{100}$$

$$D_z=0,\quad D_x=\frac{q_{40}(\delta_{33}d_{15}+\delta_{11}d_{31})(h+6z)(h-2z)}{4h^2\delta_{33}} \tag{101}$$

$$\Phi_{\mathrm{E}}=-\frac{q_{40}d_{31}(12xz^2-4hxz-h^2x-12Lx^2+4Lhz)}{4h^2\delta_{33}} \tag{102}$$

$$\begin{aligned} w=&-\frac{q_{40}(s_{11}\delta_{33}-d_{31}^2)(3L-x)x^2}{\delta_{33}h^2}\\ &-\frac{q_{40}(s_{11}\delta_{33}-d_{31}d_{33})(h-3z)(L-x)z}{\delta_{33}h^2}. \end{aligned} \tag{103}$$

We can confirm that Eqs. (100)-(103) satisfy Eqs. (98) and (99).

3.4 Fixed end–fixed end beam subjected to uniform shear force

The boundary conditions of the two longitudinal sides are

$$\begin{gathered} q_1(x)=0, \quad q_2(x)=0, \quad q_3(x)=0, \quad q_4(x)=q_{40} \\ q_1^{\mathrm{E}}(x)=0, \quad q_2^{\mathrm{E}}(x)=0. \end{gathered} \tag{104}$$

The boundary conditions of two ends of beam are

$$\begin{gathered} u\big|_{\substack{x=0\\z=0}}=0,\ w\big|_{\substack{x=0\\z=0}}=0,\ u\big|_{\substack{x=L\\z=0}}=0,\ w\big|_{\substack{x=L\\z=0}}=0 \\ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{\substack{x=0\\z=0}}=0,\ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{\substack{x=L\\z=0}}=0,\ \Phi_{\mathrm{E}}\big|_{\substack{x=0\\z=0}}=0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=0}\right)dz=0,\quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=L}\right)dz=0. \end{gathered} \tag{105}$$

The solutions are

$$\sigma_{xx}=-\frac{q_{40}(L-2x)}{2h},\quad \sigma_{zz}=0,\quad \sigma_{xz}=\frac{q_{40}(h-2z)}{2h} \tag{106}$$

$$D_z=0,\quad D_x=-\frac{q_{40}(\delta_{11}d_{31}+\delta_{33}d_{15})z}{\delta_{33}h} \tag{107}$$

$$\Phi_{\mathrm{E}}=-\frac{q_{40}(\delta_{11}d_{31}Lz-\delta_{33}d_{15}hx-2d_{31}\delta_{11}xz)}{2h\delta_{11}\delta_{33}} \tag{108}$$

$$w=-\frac{q_{40}(s_{11}\delta_{33}-d_{31}d_{33})(L-2x)z}{2\delta_{33}h}. \tag{109}$$

We can confirm that Eqs. (106)-(109) satisfy Eqs. (104) and (105).

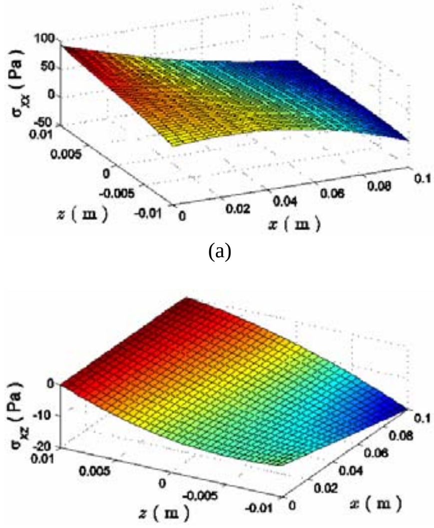
3.5 Fixed end–roller end beam subjected to linear shear force

The boundary conditions of the two longitudinal sides are

$$\begin{gathered} q_1(x)=0, \quad q_2(x)=0, \quad q_3(x)=0, \quad q_1^{\mathrm{E}}(x)=0 \\ q_4(x)=q_{40}+q_{41}\left(\frac{x}{L}\right), \quad q_2^{\mathrm{E}}(x)=0. \end{gathered} \tag{110}$$

The boundary conditions of two ends of beam are

$$\begin{gathered} u\big|_{\substack{x=0\\z=0}}=0,\ w\big|_{\substack{x=0\\z=0}}=0,\ u\big|_{\substack{x=L\\z=0}}=0 \\ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{\substack{x=0\\z=0}}=0,\ \Phi_{\mathrm{E}}\big|_{\substack{x=0\\z=0}}=0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)dz=0,\quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)zdz=0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=0}\right)dz=0,\quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=L}\right)dz=0. \end{gathered} \tag{111}$$



(a)

(b)

Fig. 2. Distribution of σ_{xx} and σ_{xz} for whole piezoelectric beam.

The numerical results for distribution of σ_{xx} and σ_{xz} in whole piezoelectric beam are given in Fig. 2(a) and (b), respectively, with loads parameters taken as $q_{41}=q_{40}=-10\,\mathrm{Pa}$.

It is noted from Fig. 2(a) the max value of σ_{xx} take place near the region of coordinate (0, 0.01). But for σ_{xz}, the max value arises near the region of $z=0.01$.

3.6 Fixed end–hinged end beam subjected to quadratic electric displacement

The boundary conditions of the two longitudinal sides are

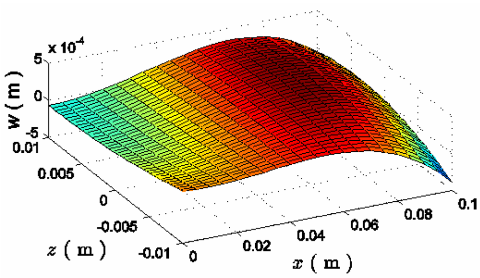
$$\begin{gathered} q_1(x)=0, \quad q_2(x)=0, \quad q_3(x)=0, \quad q_4(x)=0 \\ q_1^{\mathrm{E}}(x)=0, \quad q_2^{\mathrm{E}}(x)=q_{20}^{\mathrm{E}}+q_{21}^{\mathrm{E}}\left(\frac{x}{L}\right)+q_{22}^{\mathrm{E}}\left(\frac{x}{L}\right)^2. \end{gathered} \tag{112}$$

The boundary conditions of two ends of beam are

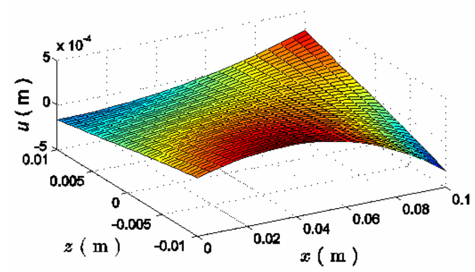
$$\begin{gathered} u\big|_{\substack{x=0\\z=0}}=0,\ w\big|_{\substack{x=0\\z=0}}=0,\ u\big|_{\substack{x=L\\z=0}}=0,\ w\big|_{\substack{x=L\\z=0}}=0 \\ \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{\substack{x=0\\z=0}}=0,\ \Phi_{\mathrm{E}}\big|_{\substack{x=0\\z=0}}=0,\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{xx}\big|_{x=L}\right)zdz=0 \\ \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=0}\right)dz=0,\quad \int_{-\frac{h}{2}}^{\frac{h}{2}}\left(D_x\big|_{x=L}\right)dz=0. \end{gathered} \tag{113}$$

The numerical results for distribution of w and u in whole piezoelectric beam are given in Fig. 3(a) and (b), respectively, with loads parameters taken as $q_{20}^{\mathrm{E}}=q_{21}^{\mathrm{E}}=q_{22}^{\mathrm{E}}=10^{-4}\,\mathrm{C/m^2}$.

We can find from Fig. 3(a) that the max value of w do not arises at the center of the beam and also not the bottom and upper surface of the beam. As to the distribution of u, the change interval of its value are the same to w, but the max value of u take place at the borderline of the beam.

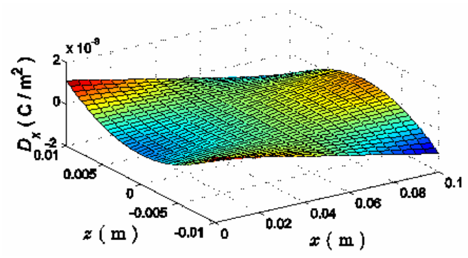


(a)

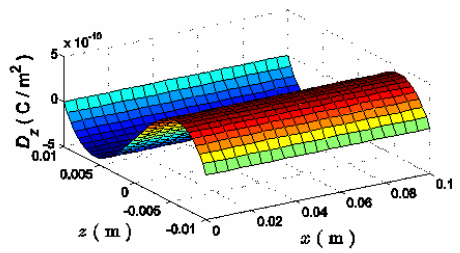


(b)

Fig. 3. Distribution of w and u for whole piezoelectric beam.



(a)



(b)

Fig. 4. Distribution of D_x and D_z for whole piezoelectric beam.

3.7 Fixed end–fixed end beam subjected to uniform pressure force

The boundary conditions of the two longitudinal sides are

$$q_1(x) = 0, \quad q_2(x) = 0, \quad q_3(x) = q_{30}, \quad q_4(x) = 0$$
$$q_1^{\mathrm{E}}(x) = 0, \quad q_2^{\mathrm{E}}(x) = 0. \tag{114}$$

The ends supported conditions are the same to Eq. (105). The numerical results for distribution of D_x and D_z in whole piezoelectric beam are given in Fig. 4(a) and (b), respectively, with loads parameters taken as $q_{30} = -10\,\mathrm{Pa}$.

Comparing Fig. 4(a) with (b), the distribution rule of D_x and D_z is totally different. The distribution of D_x can be regarded as antisymmetry with respect to the center position $(0.05, 0)$ of beam. The distribution of D_z can be treated as antisymmetry against the axis line $z = 0$ of the beam.

4. Conclusions

The formulae presented in this paper can consider piezoelectric plane beam subjected to arbitrary mechanical and electrical loads with various ends supported conditions. Comparing this general method with traditional trial-and-error method, the most advantage is it can obtain the exact solutions directly and does not need to guess and modify the form of stress function or electrical displacement function. Examples show the correctness and generalization of this method.

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Appendix

$$\begin{cases} a_{11} = \dfrac{d_{31}^2 - s_{11}\delta_{33}}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{12} = -\dfrac{(2s_{13}+s_{44})\delta_{33} + d_{31}(d_{15}-d_{33})}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{13} = -\dfrac{s_{33}\delta_{33}}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{14} = \dfrac{\delta_{33}}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \end{cases} \tag{A1}$$

$$\begin{cases} a_{21} = \dfrac{s_{11}\delta_{11} + d_{31}(d_{15}-d_{33})}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{22} = \dfrac{(2s_{13}+s_{44})\delta_{11} - (d_{15}-d_{33})^2}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{23} = \dfrac{s_{33}\delta_{11}}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \\ a_{24} = -\dfrac{\delta_{11}}{\delta_{33}(d_{15}-d_{33}) + d_{31}\delta_{11}} \end{cases} \tag{A2}$$

$$\begin{cases} a_{31} = (s_{11} - d_{31}a_{21})\lambda_1^3 + (s_{13} - d_{31}a_{22})\lambda_1 - d_{31}\dfrac{a_{23}}{\lambda_1} \\ a_{32} = (s_{11} - d_{31}a_{21})\lambda_2^3 + (s_{13} - d_{31}a_{22})\lambda_2 - d_{31}\dfrac{a_{23}}{\lambda_2} \\ a_{33} = (s_{11} - d_{31}a_{21})\lambda_3^3 + (s_{13} - d_{31}a_{22})\lambda_3 - d_{31}\dfrac{a_{23}}{\lambda_3} \end{cases} \tag{A3}$$

$$\begin{cases} a_{41} = (s_{13} - d_{33}a_{21})\lambda_1^2 + s_{33} - d_{33}a_{22} - d_{33}\dfrac{a_{23}}{\lambda_1^2} \\ a_{42} = (s_{13} - d_{33}a_{21})\lambda_2^2 + s_{33} - d_{33}a_{22} - d_{33}\dfrac{a_{23}}{\lambda_2^2} \\ a_{43} = (s_{13} - d_{33}a_{21})\lambda_3^2 + s_{33} - d_{33}a_{22} - d_{33}\dfrac{a_{23}}{\lambda_3^2} \end{cases} \tag{A4}$$

$$\begin{cases} a_{51} = -(d_{15} + \delta_{11}a_{21})\lambda_1^2 - \delta_{11}a_{22} - \delta_{11}\dfrac{a_{23}}{\lambda_1^2} \\ a_{52} = -(d_{15} + \delta_{11}a_{21})\lambda_2^2 - \delta_{11}a_{22} - \delta_{11}\dfrac{a_{23}}{\lambda_2^2} \\ a_{53} = -(d_{15} + \delta_{11}a_{21})\lambda_3^2 - \delta_{11}a_{22} - \delta_{11}\dfrac{a_{23}}{\lambda_3^2} \end{cases} \tag{A5}$$

$$\begin{cases} a_{61} = (d_{31} - \delta_{33}a_{21})\lambda_1^3 + (d_{33} - \delta_{33}a_{22})\lambda_1 - \delta_{33}\dfrac{a_{23}}{\lambda_1} \\ a_{62} = (d_{31} - \delta_{33}a_{21})\lambda_2^3 + (d_{33} - \delta_{33}a_{22})\lambda_2 - \delta_{33}\dfrac{a_{23}}{\lambda_2} \\ a_{63} = (d_{31} - \delta_{33}a_{21})\lambda_3^3 + (d_{33} - \delta_{33}a_{22})\lambda_3 - \delta_{33}\dfrac{a_{23}}{\lambda_3} \end{cases} \tag{A6}$$

$$\begin{cases} a_{71} = a_{21}\lambda_1^2 + a_{22} + \dfrac{a_{23}}{\lambda_1^2} \\ a_{72} = a_{21}\lambda_2^2 + a_{22} + \dfrac{a_{23}}{\lambda_2^2} \\ a_{73} = a_{21}\lambda_3^2 + a_{22} + \dfrac{a_{23}}{\lambda_3^2} \end{cases} \tag{A7}$$

$$\begin{cases} a_{81}^1 = a_{80}^1 - \dfrac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)}, \quad a_{82}^1 = a_{80}^1 - \dfrac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1+\lambda_2)} \\ a_{84}^1 = a_{80}^1 - \dfrac{\lambda_3(\lambda_1+\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)}, \quad a_{83}^1 = a_{80}^1 - \dfrac{\lambda_3(\lambda_1-\lambda_3)}{\lambda_2(\lambda_1-\lambda_2)} \\ a_{80}^1 = \dfrac{a_{63}\lambda_1 - a_{61}\lambda_3}{a_{62}\lambda_1 - a_{61}\lambda_2}, \quad a_{85}^1 = \dfrac{2\lambda_2}{\lambda_1+\lambda_2}, \quad a_{86}^1 = \dfrac{2\lambda_2}{\lambda_1-\lambda_2} \end{cases} \tag{A8}$$

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